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ADDENDUM

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Fan Hong-yi

CCAST (World Laboratory) PO Box 8730, Beijing 100080, People's Republic of China, and (mailing address) Department of Material Science and Engineering, China University of Science and Technology, Hefei, Anhui 230026, People's Republic of China

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Abstract. The properties of Weyl ordered products of operators are investigated and the technique of integration within Weyl ordered product (IWWP) is introduced. The overcompleteness relation of the coherent state is then recast into Weyl ordered form. In so doing, a new approach for Weyl ordering quantum mechanical operators is presented.

1. Introduction

In [1] a convenient approach for normally ordering some multimode exponential operators is developed, which is based on the technique of integration within normally ordered products of operators (IWNP). Then in [2] we use the technique of integration within antinormally ordered products to present a formalism for antinormally ordering exponential operators. There also exists another well known operator ordering rule—Weyl ordering—which is closely related to the Weyl quantization rule [3]

$$F(P, Q) = \iint_{-\infty}^{\infty} \mathrm{d}p\, \mathrm{d}q\, f(p, q)\Delta(p, q) \tag{1}$$

where the quantum operators, F, P and Q correspond to the classical quantities f, p and q, respectively, and $\Delta(p, q)$ is the Wigner operator [4], defined as

$$\begin{aligned}\Delta(p, q) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}v\, \mathrm{e}^{-\mathrm{i}pv}\left|q-\frac{v}{2}\right\rangle\left\langle q+\frac{v}{2}\right| \\ &= \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \mathrm{d}u\, \mathrm{d}v\, \exp[\mathrm{i}(q-Q)v+\mathrm{i}(p-P)u]\end{aligned} \tag{2}$$

where $[Q, P] = \mathrm{i}$, $\hbar = 1$, $Q|q\rangle = q|q\rangle$. The inverse correspondence of (1) is given by [3, 4]

$$f(p, q) = 2\pi\, \mathrm{Tr}[F(P, Q)\Delta(p, q)]. \tag{3}$$

For example, the Weyl correspondence operator of $q^m p^r$ is [5]

$$q^m p^r \to (\tfrac{1}{2})^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} Q^{m-l} P^r Q^l \tag{4}$$

which is in Weyl ordering. An interesting question thus naturally arises: can we find a convenient approach for Weyl ordering quantum mechanical operators (especially

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convenient for operators in Fock space)? The answer is affirmative. In section 2 we first introduce the properties of Weyl ordered products of operators, then in section 3 we recast the overcompleteness relation of the coherent state [6] into Weyl ordered form by using the technique of integration within Weyl ordered product (IWWP). On the basis of sections 2 and 3 we provide an approach for Weyl ordering quantum operators. Some examples showing how Weyl ordered operators can be easily derived are given in section 4.

2. Weyl ordered product of operators and the IWWP technique

Let the symbol $\vdots\,\vdots$ denote the Weyl ordered product; we rewrite (1) as

$$\vdots f(P,Q)\vdots=\iint_{-\infty}^{\infty}\mathrm{d}p\,\mathrm{d}q\,f(p,q)\Delta(p,q) \tag{5}$$

which indicates the classical correspondence of a Weyl ordered operator $\vdots f(P,Q)\vdots$ can be readily obtained by simply replacing $Q\to q$, $P\to p$. For instance, (4) represents

$$\begin{aligned}\left(\frac{1}{2}\right)^m\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}Q^{m-l}P^rQ^l &\\ =\left(\frac{1}{2}\right)^m\vdots\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}Q^{m-l}P^rQ^l\vdots &\\ =\iint_{-\infty}^{\infty}\mathrm{d}p\,\mathrm{d}q\left(\frac{1}{2}\right)^m\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}q^mp^r\Delta(p,q) &\\ =\iint_{-\infty}^{\infty}\mathrm{d}p\,\mathrm{d}q\,q^mp^r\Delta(p,q). &\end{aligned} \tag{6}$$

Using $Q=(a+a^\dagger)/\sqrt{2}$, $P=(a-a^\dagger)/\sqrt{2}\mathrm{i}$, where $a^\dagger$ and a are creation and annihilation operators in Fock space, satisfying $(a,a^\dagger)=1$, we can express (5) and (2) as

$$\vdots G(a,a^\dagger)\vdots=2\int\mathrm{d}^2\alpha\,G(\alpha,\alpha^*)\Delta(\alpha,\alpha^*)\qquad G(\alpha,\alpha^*)=f(p,q)$$
$$\alpha=(q+\mathrm{i}p)/\sqrt{2} \tag{7}$$

$$\Delta(\alpha,\alpha^*)=(2\pi^2)^{-1}\int\mathrm{d}^2z\exp[z(a^\dagger-\alpha^*)-z^*(a-\alpha)]. \tag{8}$$

The explicit form of $\Delta(\alpha,\alpha^*)$ can be derived by using the IWNP technique [7]

$$\begin{aligned}\Delta(\alpha,\alpha^*)&=(2\pi^2)^{-1}\int\mathrm{d}^2z\,{:}\exp[-\tfrac{1}{2}|z|^2+z(a^\dagger-\alpha^*)-z(a-\alpha)]{:}\\ &=\pi^{-1}{:}\exp[-2(a^\dagger-\alpha^*)(a-\alpha)]{:}\end{aligned} \tag{9}$$

which is in normal ordering. As in [8], we list the properties of the Weyl ordered product of operators:

(i) The order of Bose operators within a Weyl ordered product can be permuted.
(ii) C numbers can be taken out of the symbol $\vdots\,\vdots$ as one wishes.
(iii) The symbol $\vdots\,\vdots$ which is within another symbol $\vdots\,\vdots$ can be deleted.
(iv) A Weyl ordered product can be integrated or differentiated with respect to a C-number provided the integration is convergent.

(v) From the above and equation (5) we conclude that the Weyl ordered form of $\Delta(p, q)$ is

$$\Delta(p, q) = \vdots\delta(p-P)\delta(q-Q)\vdots$$

or

$$\Delta(\alpha, \alpha^*) = \tfrac{1}{2}\vdots\delta(\alpha - a)\delta(\alpha^* - a^\dagger)\vdots = \tfrac{1}{2}\vdots\delta(a^\dagger - \alpha^*)\delta(a - \alpha)\vdots\,. \tag{10}$$

Consequently, (5) and (7) respectively become

$$\vdots f(P, Q)\vdots = \vdots\iint_{-\infty}^{\infty} \mathrm{d}p\,\mathrm{d}q\, f(p, q)\delta(p-P)\delta(q-Q)\vdots \tag{11}$$

$$\vdots G(a, a^\dagger)\vdots = \int \mathrm{d}^2\alpha \vdots G(\alpha, \alpha^*)\delta(\alpha - a)\delta(\alpha^* - a^\dagger)\vdots\,. \tag{12}$$

Taking the correspondence (6) for example, we have

$$\iint_{-\infty}^{\infty} \mathrm{d}p\,\mathrm{d}q\, q^m p^r \vdots\delta(p-P)\delta(q-Q)\vdots = \vdots Q^m P^r\vdots\,.$$

Note that before we move $\vdots\;\vdots$ away from $\vdots Q^m P^r\vdots$ we must arrange $\vdots Q^m P^r\vdots$ as

$$\vdots\left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} Q^{m-l} P^r Q^l\vdots.$$

3. Weyl ordered form of the overcompleteness relation of coherent state

In [1] and [2] we have given the normal product form and the antinormal product form of the coherent state's overcompleteness relation, respectively. Let $|z\rangle$ be the coherent state, $|z\rangle = \exp(-\frac{1}{2}|z|^2 + za^\dagger)|0\rangle$. With use of (3) we can calculate the classical correspondence of the projection operator $|z\rangle\langle z|$, e.g.

$$\begin{aligned} 2\pi\,\mathrm{Tr}[|z\rangle\langle z|\Delta(\alpha, \alpha^*)] &= 2\langle z|\,{:}\exp[-2(a^\dagger - \alpha^*)(a - \alpha)]{:}\,|z\rangle \\ &= 2\exp[-2(z^* - \alpha^*)(z - \alpha)] \end{aligned} \tag{13}$$

where the normal product form of $\Delta(\alpha, \alpha^*)$ is exploited. As a result of (12) and (13) we obtain the Weyl ordered form of $|z\rangle\langle z|$

$$\begin{aligned} |z\rangle\langle z| &= 2\int \mathrm{d}^2\alpha\, \exp[-2(z^* - \alpha^*)(z - \alpha)]\vdots\delta(a^\dagger - \alpha^*)\delta(a - \alpha)\vdots \\ &= 2\vdots\exp[-2(z^* - a^\dagger)(z - a)]\vdots\,. \end{aligned} \tag{14}$$

Thus we may recast the overcompleteness relation of $|z\rangle$ into Weyl ordering

$$\int \frac{\mathrm{d}^2 z}{\pi} |z\rangle\langle z| = 2\int \frac{\mathrm{d}^2 z}{\pi} \vdots \mathrm{e}^{-2(z^* - a^\dagger)(z - a)}\vdots = 1 \tag{15}$$

where we have used the mathematical formula

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp[\lambda|z|^2 + fz + gz^*] = -\frac{1}{\lambda}\exp\left[-\frac{fg}{\lambda}\right] \qquad \mathrm{Re}\,\lambda < 0.$$

4. An approach for Weyl ordering quantum-mechanical operators

Using (15) we can put the Glauber P-representation of ρ [6] into Weyl ordered form

$$\rho=\int\frac{d^2z}{\pi}P(z)|z\rangle\langle z|=2\int\frac{d^2z}{\pi}P(z)\vdots e^{-2(z^*-a^\dagger)(z-a)}\vdots. \tag{16}$$

Once the P-representation $P(z)$ of a given operator is known, we can obtain its Weyl ordered form by performing the integration (16) with the help of IWWP (NOTE that $P(z)$ is directly determined by the operator's antinormal product form). Taking $e^{\lambda a^\dagger a}$ for instance, according to the antinormal product expansion $e^{\lambda a^\dagger a}=e^{-\lambda}\vdots\exp[(1-e^{-\lambda})a^\dagger a]\vdots$ and (16) we have

$$\begin{aligned}e^{\lambda a^\dagger a}&=\int\frac{d^2z}{\pi}e^{-\lambda}\exp[(1-e^{-\lambda})|z|^2]|z\rangle\langle z|\\&=2\,e^{-\lambda}\int\frac{d^2z}{\pi}\vdots\exp\{-(1+e^{-\lambda})|z|^2+2z^*a+2za^\dagger-2a^\dagger a\}\vdots\\&=\frac{2}{e^\lambda+1}\vdots\exp\left[\frac{2(e^\lambda-1)}{e^\lambda+1}a^\dagger a\right]\vdots.\end{aligned} \tag{17}$$

The second example is to put $\exp(fa^2)\exp(ga^{\dagger 2})$ into Weyl ordering, from (16) we have

$$\begin{aligned}e^{fa^2}e^{ga^{\dagger 2}}&=2\int\frac{d^2z}{\pi}\vdots\exp[-2(z^*-a^\dagger)(z-a)+fz^2+gz^{*2}]\vdots\\&=(1-fg)^{-1/2}\vdots\exp\{(1-fg)^{-1}[(fa^2+ga^{\dagger 2})+2fga^\dagger a]\}\vdots.\end{aligned} \tag{18}$$

As the last example we consider the rotation operator $\exp(-iJ_y\theta)$, where the angular momentum operator is represented by Bose operator $J_y=(1/2i)(a^\dagger b-b^\dagger a)$ (so-called Schwinger boson realization) with b and $b^\dagger$ satisfying $[b,b^\dagger]=1$, $b|\gamma\rangle=\gamma|\gamma\rangle$ (we will let $G(\beta;\beta^*)$ denote the classical correspondence of $G(b,b^\dagger)$). The antinormal product form of $e^{-iJ_y\theta}$ has been derived in [8], e.g.

$$e^{-iJ_y\theta}=\vdots\exp\left[2\sin^2\frac{\theta}{4}(a^\dagger a+b^\dagger b)-\sin\frac{\theta}{2}(a^\dagger b-b^\dagger a)\right]\vdots. \tag{19}$$

Substituting (19) into the two-mode generalization expression of (16), we have

$$\begin{aligned}e^{-iJ_y\theta}&=\int\frac{d^2z\,d^2\gamma}{\pi^2}\exp\left[2\sin^2\frac{\theta}{4}(|z|^2+|\gamma|^2)-\sin\frac{\theta}{2}(z^*\gamma-\gamma^*z)\right]|z\gamma\rangle\langle z\gamma|\\&=4\int\frac{d^2z\,d^2\gamma}{\pi^2}\vdots\exp\left\{-2(|z|^2+|\gamma|^2)\cos^2\frac{\theta}{4}+\sin\frac{\theta}{2}(\gamma^*z-z^*\gamma)\right.\\&\quad\left.+2(z^*a+za^\dagger+\gamma^*b+b^\dagger\gamma-b^\dagger b-a^\dagger a)\right\}\vdots\\&=\sec^2\frac{\theta}{4}\vdots\exp\left[2(ab^\dagger-ba^\dagger)\tan\frac{\theta}{4}\right]\vdots.\end{aligned} \tag{20}$$

which is the Weyl ordered form of $\exp(-\mathrm{i}J_y\theta)$. From (7) we can directly obtain the classical correspondence of $\mathrm{e}^{-\mathrm{i}J_y\theta}$

$$\mathrm{e}^{-\mathrm{i}J_y\theta} \to \sec^2\frac{\theta}{4}\exp\left[2\tan\frac{\theta}{4}(\alpha\beta^*-\beta\alpha^*)\right]$$
$$=\sec^2\frac{\theta}{4}\exp\left[2\mathrm{i}(q'p-p'q)\tan\frac{\theta}{4}\right] \qquad \left(\beta\equiv\frac{1}{\sqrt{2}}(q'+\mathrm{i}p')\right).$$

In summary, the present work together with [1, 2] have provided convenient approaches to Weyl ordering, normally ordering and antinormally ordering operators.

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